PI is not at least as succinct as MODS

Nikolay Kaleyski

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"A Knowledge Compilation Map", Adnan Darwiche & Pierre Marquis (2002)

L	NNF	DNNF	d-DNNF	sd-DNNF	FBDD	OBDD	$OBDD_<$	DNF	CNF	PI	IP	MODS
NNF	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$
DNNF	$\not\leq^*$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\not\leq^*$	?	$\leq$	$\leq$
d-DNNF	$\not\leq^*$	$\not\leq^*$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\not\leq^*$	$\not\leq^*$	?	?	$\leq$
sd-DNNF	$\not\leq^*$	$\not\leq^*$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\not\leq^*$	$\not\leq^*$	?	?	$\leq$
FBDD	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\leq$	$\leq$	$\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\leq$
OBDD	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\leq$	$\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\leq$
$OBDD_<$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\leq$
DNF	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\leq$	$\not\leq$	$\not\leq$	$\leq$	$\leq$
CNF	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\leq$	$\leq$	$\not\leq$	$\leq$
PI	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\leq$	$\not\leq$	?
IP	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\leq$	$\leq$
MODS	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\not\leq$	$\leq$

Known results in knowledge compilation (continued)

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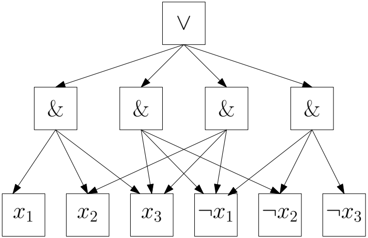
Background: Sentences and Formulas

- A **sentence** is a directed acyclic graph with Boolean operations in the internal nodes and literals in the leaves.



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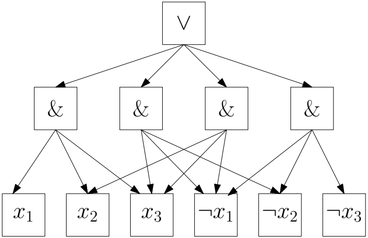

- Every sentence has an **equivalent Boolean formula**.

$$x_1x_2x_3 \vee \overline{x_1x_2}x_3 \vee \overline{x_1}x_2x_3 \vee \overline{x_1x_2x_3}$$

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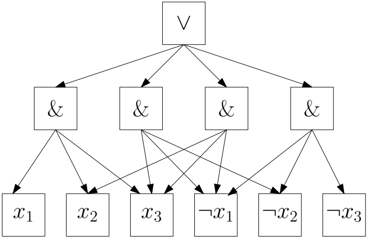

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- We will work with formulas.

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- A language L_1 is **at least as succinct** as a language L_2 ($L_1 \leq L_2$) if there is a polynomial p such that

$$(\forall \varphi_2 \in L_2)(\exists \varphi_1 \in L_1)(\varphi_1 \equiv \varphi_2 \ \& \ |\varphi_1| \leq p(|\varphi_2|))$$

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$$x_1\overline{x_2x_3}x_4x_5$$

for

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- The MODS language is not at least as succinct as PI as witnessed by

$$\Sigma = \bigvee_{i=1}^{n} x_i$$

Overview

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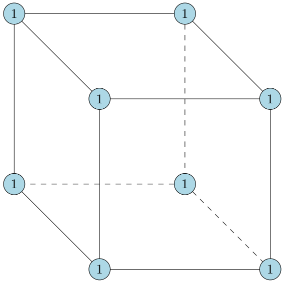
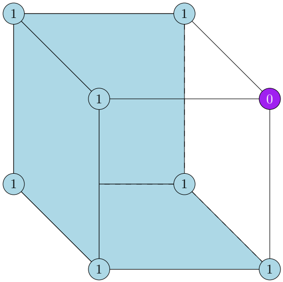
- Thesis available at *Charles University's Thesis Repository*.

Finding a counterexample

- Sequence of Boolean functions φ_i with "many" prime implicates and few models...





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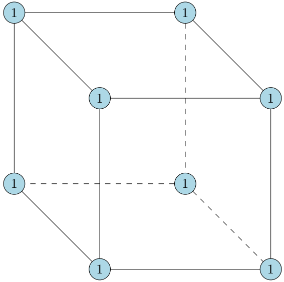

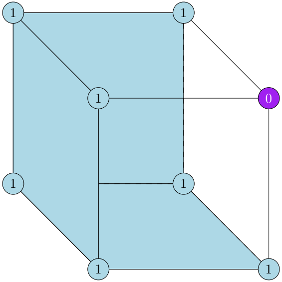

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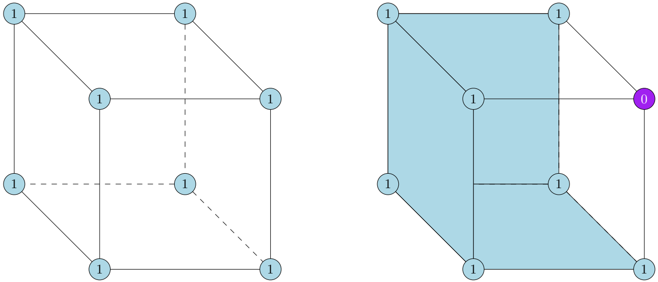
- Geometric view: inserting false points into a hypercube



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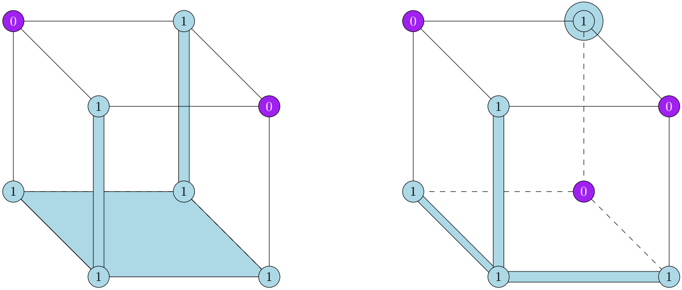

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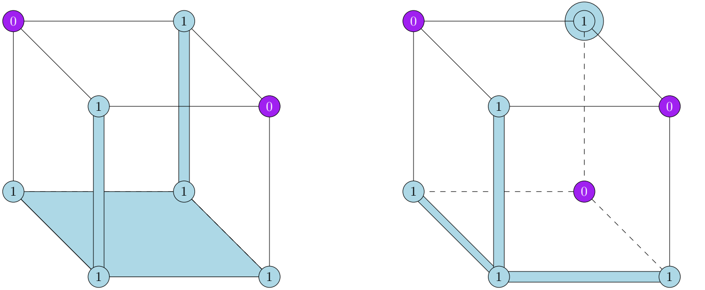

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- Suggestion: linear code

Construction

- A sequence of matrices $\{\mathbf{A}_i\}_{i \in \mathcal{N}}$ is defined as

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$$\mathbf{A}_{i+1} = \begin{pmatrix} \mathbf{A}_i & \mathbf{A}_i \\ \mathbf{A}_i & \overline{\mathbf{A}}_i \end{pmatrix}$$

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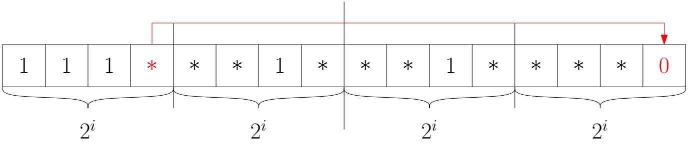
- It is shown that φ_i has many prime *implicants*; then its negation $\overline{\varphi}_i$ has many prime *imlicates*.

Number of prime implicants: Lower bound

- Given a prime implicant p of φ_i with a single negative literal whose positive part agrees with precisely one row of $\mathbf{B}_i$, we can construct 2^i different prime implicants of φ_{i+2} via the following construction step:



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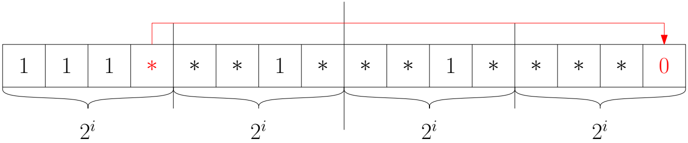

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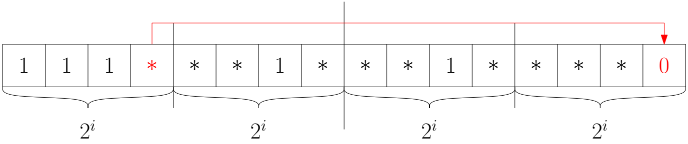

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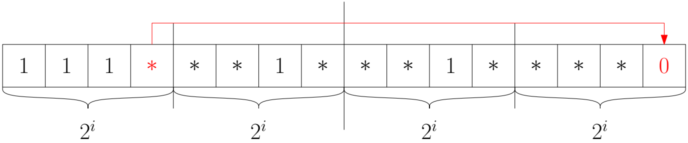

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- Hence $\overline{\varphi_i}$ has "many" prime *implicates* w.r.t. to its number of *true* points, or models.

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- Fixing the values of variables has "global effects" and affects other variables as well:

1	1	*	1	*	*	*	*	*	*	*	*	*	*	*	*
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
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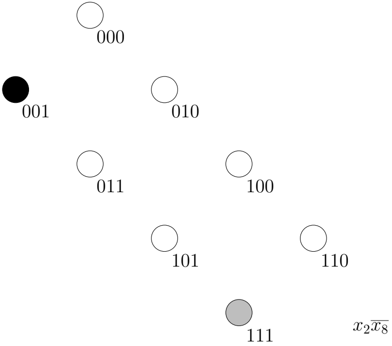

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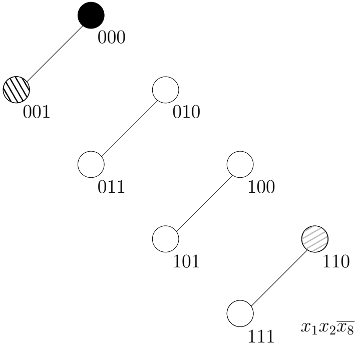

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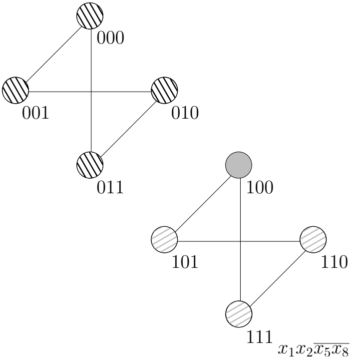

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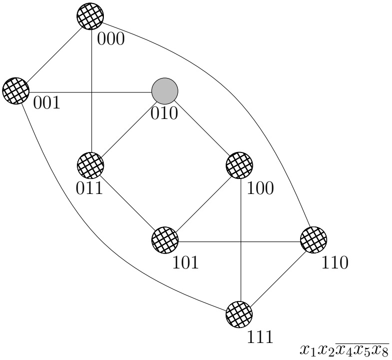

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- Thus no prime implicant of φ_i can have more than $(i + 2)$ literals, and their number is at most

$$\sum_{l=1}^{i+2} \binom{n}{l} 2^l \leq 3^{n+2} \in \Theta(3^n)$$

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- After $2 + i$ fixations, the graph becomes connected and no further non-redundant fixations can be performed.
- Thus no prime implicant of φ_i can have more than $(i + 2)$ literals, and their number is at most

$$\sum_{l=1}^{i+2} \binom{n}{l} 2^l \leq 3^{n+2} \in \Theta(3^n)$$

- The number of false points of φ_n is exponential in n!

Number of prime implicants: Exact formula

- The polar graphs are analyzed further by observing a correspondence between variable fixations yielding connected graphs and generating sets of the vector space $\{0,1\}^i$.

Number of prime implicants: Exact formula

- The polar graphs are analyzed further by observing a correspondence between variable fixations yielding connected graphs and generating sets of the vector space $\{0,1\}^i$.
- The exact number of prime implicants of φ_i is

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{n+2i+1} \cdot \alpha_n^{2i}}{(2i+1)(2i+2)}$$

where α_n^i is the number of i-element linearly independent sets in $\{0,1\}^n$.